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A new uniqueness criterion for the number of periodic orbits of Abel equations

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Abstract

A solution of the Abel equation $\dot{x} = A(t)x^3 + B(t)x^2$ such that $x(0) = x(1)$ is called a periodic orbit of the equation. Our main result proves that if there exist two real numbers a and b such that the function $aA(t) + bB(t)$ is not identically zero, and does not change sign in [0, 1] then the Abel differential equation has at most one non-zero periodic orbit. Furthermore, when this periodic orbit exists, it is hyperbolic. This result extends the known criteria about the Abel equation that only refer to the cases where either $A(t) \not\equiv 0$ or $B(t) \not\equiv 0$ does not change sign. We apply this new criterion to study the number of periodic solutions of two simple cases of Abel equations: the one where the functions $A(t)$ and $B(t)$ are 1-periodic trigonometric polynomials of degree one and the case where these two functions are polynomials with three monomials. Finally, we give an upper bound for the number of isolated periodic orbits of the general Abel equation $\dot{x} = A(t)x^3 + B(t)x^2 + C(t)x$, when $A(t)$, $B(t)$ and $C(t)$ satisfy adequate conditions.

Keywords: Abel equation; Periodic orbit

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1. Introduction and main results

In this work we consider smooth Abel equations of the form

$$\dot{x} = \frac{dx}{dt} = A(t)x^3 + B(t)x^2, \tag{1}$$

defined on the strip $\mathcal{S} = \{(t, x)\colon\ t \in [0, 1],\ x \in \mathbb{R}\}$. For these equations we study the number of solutions which start on the line $t = 0$ at some point $x = x_0$ and arrive until the line $t = 1$ having the same height, namely $x = x_0$. These solutions will be called for short, *periodic orbits*. Note that $x = 0$ is always a periodic orbit of the equation.

Observe that for the Abel equation with A and B being 1-periodic functions, Eq. (1) is indeed a differential equation defined on a cylinder, and this type of *periodic orbits* are actual periodic orbits of the Abel equation. In particular, the trigonometric case is important because some families of differential systems on the plane can be transformed, after an adequate change of variables, into this type of Abel equation, see [4,10]. Thus, the criteria that we obtain in this paper to control the number of periodic orbits of Abel equations can also be used to obtain upper bounds of the number of limit cycles of several families of planar polynomial vector fields.

As usual, we will say that a periodic orbit of Eq. (1) is *hyperbolic* if the Poincaré map between $t = 0$ and $t = 1$ has derivative different from one at the initial condition of the periodic orbit. It is not difficult to check that $x = 0$ is always a non-hyperbolic periodic orbit of Eq. (1). We also will say that Eq. (1) has a *center* at a given periodic orbit if there exists a neighborhood of this solution where all the orbits are periodic.

To our knowledge the more general results that give upper bounds of the number of periodic orbits of a complete Abel equation of the form $dx/dt = A(t)x^3 + B(t)x^2 + C(t)x$, are the ones given in [6,8,9] and [12, Theorem 9.7]. As we will see in Proposition 8, these results applied to the case $C(t) \equiv 0$ give the following result: if either $A(t) \not\equiv 0$ or $B(t) \not\equiv 0$ does not change sign in $[0, 1]$ then the maximum number of non-zero periodic orbits of Eq. (1) is one. Furthermore, when this periodic orbit exists it is hyperbolic. Our main theorem extends these results giving a new criterion of uniqueness of non-zero periodic orbits of Eq. (1).

Theorem A. *Consider the Abel equation* (1). *Assume that there exist two real numbers a and b such that $aA(t) + bB(t)$ does not vanish identically and does not change sign in* $[0, 1]$. *Then it has at most one non-zero periodic orbit. Furthermore, when this periodic orbit exists, it is hyperbolic.*

From the proof of the above theorem we can also obtain information about the location of the non-zero periodic orbit and in some cases prove that this periodic orbit actually exists, see Remark 13.

In Section 5 we extend the above theorem to the more general Abel equation

$$\frac{dx}{dt} = A(t)x^3 + B(t)x^2 + C(t)x, \tag{2}$$

when $\int_0^1 C(t)\,dt = 0$. We also prove similar, but weaker, results when this last equality does not hold, see Theorem 16.

It is well-known that trigonometrical Abel equations of the form (1) can have an arbitrary number of isolated periodic orbits and that this number increases with the degree of the trigonometrical polynomials $A(t)$ and $B(t)$; see for instance [8] or [11]. For this reason Lins in [8] and Il'yashenko in [7] have proposed to study the problem of giving explicit and realistic bounds for general Abel equations in terms of the degrees of A and B. As an application of Theorem A we consider the simple case of trigonometrical polynomials of degree one. The case where one of the functions has degree zero is much easier and it is totally solved in Section 2, see Remark 9.

Theorem B. *Consider the Abel equation*

$$\frac{dx}{dt} = \big(a_0 + a_1\cos(2\pi t) + a_2\sin(2\pi t)\big)x^3 + \big(b_0 + b_1\cos(2\pi t) + b_2\sin(2\pi t)\big)x^2,$$

being a_0, a_1, a_2, b_0, b_1 and b_2 arbitrary real numbers.

(a) *It has a center at $x = 0$ if and only if $a_0 = b_0 = a_2b_1 - a_1b_2 = 0$.*
(b) *If it has not a center at $x = 0$ and one of the conditions $a_0^2 \geqslant a_1^2 + a_2^2$, $b_0^2 \geqslant b_1^2 + b_2^2$, or $(a_2b_0 - a_0b_2)^2 + (a_0b_1 - a_1b_0)^2 \geqslant (a_2b_1 - a_1b_2)^2$ is satisfied then it has at most one non-zero periodic orbit. Furthermore, when this periodic orbit exists, it is hyperbolic.*
(c) *There are equations of the above form having at least two non-zero hyperbolic periodic orbits.*

On the other hand Abel equations having A and B polynomials in the t-variable are also considered in the literature, see for instance [1,3] (and references therein) or [8]. In particular in [3], by using Bautin's approach and the notion of Bautin ideal, the authors study the periodic orbits bifurcating from $x = 0$ as well as the center conditions in case where A and B are polynomials of degrees less than 2. These conditions coincide with the ones given in part (a) of the theorem bellow in the special case $j = 1, k = 2$. We consider the more general situation where A and B are polynomials with three monomials. From Theorem A we obtain the following result.

Theorem C. *Consider the Abel equation*

$$\frac{dx}{dt} = \big(a_0 + a_1t^j + a_2t^k\big)x^3 + \big(b_0 + b_1t^j + b_2t^k\big)x^2, \tag{3}$$

with $j, k \in \mathbb{N}$, $0 < j < k$, and a_0, a_1, a_2, b_0, b_1 and b_2 arbitrary real numbers.

(a) *It has a center at $x = 0$ if and only if $a_0 + a_1/(j+1) + a_2/(k+1) = b_0 + b_1/(j+1) + b_2/(k+1) = a_2b_1 - a_1b_2 = 0$.*
(b) *If it has not a center at $x = 0$, consider the following sets of conditions*:
   (1) *either $a_2b_1 - a_1b_2 = 0$ or $\frac{a_2b_0 - a_0b_2}{a_2b_1 - a_1b_2} \notin (-1, 0)$,*
   (2) *either $a_1b_2 - a_2b_1 = 0$ or $\frac{a_1b_0 - a_0b_1}{a_1b_2 - a_2b_1} \notin (-1, 0)$,*
   (3) *either $a_0b_2 - a_2b_0 = 0$ or $\frac{a_0b_1 - a_1b_0}{a_0b_2 - a_2b_0} \notin (-1, 0)$.*
   If one of the above conditions is satisfied, then it has at most one non-zero periodic orbit. Furthermore, when this periodic orbit exists, it is hyperbolic.
(c) *There are equations of the form* (3) *having at least two non-zero hyperbolic periodic orbits.*

Notice that the above two theorems do not solve completely the problem of bounding the number of periodic orbits for the considered Abel equations. Both results prove the existence of at most one non-zero hyperbolic periodic orbit, but under some hypotheses. On the other hand, in both cases, the maximum number of non-zero periodic orbits that we have been able to obtain is two.

Finally, consider the case where the vector field associated to Eq. (3) coincide in both boundaries of $\mathcal{S}$, i.e. when $A(0) = A(1)$ and $B(0) = B(1)$. It is clear that in this case there are examples of (3) having at least one non-zero periodic orbit: it suffices to take for instance $a_1 = a_2 = b_1 = b_2 = 0$ and $a_0 b_0 \neq 0$. We have the following corollary which solves the problem of the number of periodic solutions of this differential equation.

Corollary 1. *Consider Eq.* (3) *with the functions A and B satisfying $A(0) = A(1)$ and $B(0) = B(1)$, i.e. $a_1 + a_2 = b_1 + b_2 = 0$, and not having a center at $x = 0$. Then it has at most one non-zero periodic orbit. Furthermore, when this periodic orbit exists, it is hyperbolic.*

2. Preliminary results

In this section we present some preliminary results. The first one is a generalization of the well-known result that gives the stability of a periodic orbit. The second one adapts the ideas presented in [5] to give upper bounds for the number of periodic orbits for some planar differential equations to Abel equations.

Consider a system

$$\begin{cases} \dot{x} = P(x, y), \\ \dot{y} = Q(x, y) \end{cases} \tag{4}$$

of class $N \geqslant 1$, and let L be the piece of the orbit associated to a solution $(x, y) = (\varphi(t), \psi(t))$ for $t \in [0, \tau]$. Assume that through the two boundary points of L there are defined two transversal curves Σ_0 and Σ_τ. By using the continuous dependence with respect to initial conditions, the flow of system (4) defines a Poincaré map Π between Σ_0 and Σ_τ. To study the derivative of this map at $L \cap \Sigma_0$ let us parametrize both sections. Given some small enough $\varepsilon > 0$, consider, without loss of generality, $\Sigma_t = \{(x, y) = (\alpha(t, n), \beta(t, n)) \colon |n| < \varepsilon\}$ for $t = 0$ and $t = \tau$, respectively. Furthermore, assume that $(\alpha(t, 0), \beta(t, 0)) \in L$ and

$$\Delta(0, 0)\Delta(\tau, 0) > 0,$$

where

$$\Delta(t, 0) = \begin{vmatrix} \varphi'(t) & \psi'(t) \\ \frac{\partial \alpha}{\partial n}(t, 0) & \frac{\partial \beta}{\partial n}(t, 0) \end{vmatrix}. \tag{5}$$

Following the steps of the study of the stability of a periodic orbit given in [2, Section 13] we obtain the following result.

Theorem 2. *Let Π be the Poincaré map between two transversal sections Σ_0 and Σ_τ of an orbit $L = \{(\varphi(t), \psi(t)) \colon t \in [0, \tau]\}$ of* (4)*. Then the derivative of Π at $p \in L \cap \Sigma_0$ is given by*

$$\Pi'(p) = \frac{\Delta(0,0)}{\Delta(\tau,0)} \exp\left(\int_0^\tau \left(\frac{\partial P}{\partial x}\big(\varphi(s),\psi(s)\big) + \frac{\partial Q}{\partial y}\big(\varphi(s),\psi(s)\big) \right) ds \right),$$

where the function Δ *is given in* (5)*.*

Note that if L is a periodic orbit of (4) and τ is its period then $\Delta(0,0) = \Delta(\tau,0)$ in the above formula, giving rise to the well-known formula for knowing the hyperbolicity of a planar periodic orbit.

In the following we will apply the above result to Abel equations to determine upper bounds for the number of periodic orbits that they can have on the strip $\mathcal{S}$.

Corollary 3. *Let* $X(t,x)$ *be the* $\mathcal{C}^1$ *vector field associated to the system*

$$\dot{t} = 1, \qquad \dot{x} = A(t)x^3 + B(t)x^2 + C(t)x, \tag{6}$$

on the strip $\mathcal{S}$*. Let* $x = \gamma(t)$ *be a solution of* (6) *defined in* $[0,1]$*. Then, for any non-zero* $\mathcal{C}^1$ *function* $g(t,x)$ *the derivative of the Poincaré map between* $x = 0$ *and* $x = 1$ *at* $(0,\gamma(0))$ *is given by*

$$\Pi'\big((0,\gamma(0))\big) = \frac{|g(0,\gamma(0))|}{|g(1,\gamma(1))|} \exp\left(\int_0^{\tau_g} \operatorname{div}\Big(\big|g\big(\gamma\big(t(s)\big)\big)\big| X\big(\gamma\big(t(s)\big)\big)\Big)\, ds \right),$$

being τ_g *a positive number and* $t(s)$ *an increasing function, both depending on* g *and given in the proof.*

Proof. Since $g(t,x)$ does not vanish on $\mathcal{S}$ system (6) is equivalent to

$$\begin{cases} t' = dt/ds = |g(t,x)|, \\ x' = dx/ds = |g(t,x)|(A(t)x^3 + B(t)x^2 + C(t)x). \end{cases} \tag{7}$$

Let $(t,\gamma(t))$ be a solution of (6). Then there exists a function $t(s)$ and a positive number τ_g such that $(t(s),\gamma(t(s)))$ is also a solution of (7), for $s \in [0,\tau_g]$. We take as transversal sections the two borders of the strip $\mathcal{S}$, parametrized as $(\alpha(s,n),\beta(s,n)) = (t(s),\gamma(t(s)) + n)$ for $s = 0, \tau_g$. Then, applying Theorem 2 we obtain

$$\Pi'\big((0,\gamma(0))\big) = \frac{\Delta(0,\gamma(0))}{\Delta(1,\gamma(1))} \exp\left(\int_0^{\tau_g} \operatorname{div}\Big(\big|g\big(t(s),\gamma\big(t(s)\big)\big)\big| X\big(t(s),\gamma\big(t(s)\big)\big)\Big)\, ds \right),$$

with

$$\Delta(t,x) = \begin{vmatrix} |g(t,x)| & (A(t)x^3 + B(t)x^2 + C(t)x)|g(t,x)| \\ 0 & 1 \end{vmatrix},$$

for $t = 0, 1$ and then, the result follows. $\square$

Applying the previous corollary we get the next result.

Corollary 4. *Consider the Abel equation*

$$\frac{dx}{dt} = A(t)x^3 + B(t)x^2 + C(t)x := h(t,x), \tag{8}$$

on the strip $\mathcal{S}$ and let $g(t,x)$ be a non-zero $\mathcal{C}^1$ function, 1-periodic in t. Let K be a connected region where $\operatorname{div}(|g(t,x)|(1,h(t,x)))$ does not change sign, and vanishes only on a null measure Lebesgue set, which is not invariant by the flow of Eq. (8). Then (8) has, at most, one periodic orbit completely contained on K. Moreover, if it exists, it is hyperbolic and its stability is given by the sign of $\operatorname{div}(|g(t,x)|(1,h(t,x)))$.

Proof. Let $x=\gamma(t)$ be a 1-periodic orbit of Eq. (8). Then $(t(s),\gamma(t(s)))$ is a solution of (7), defined for $s\in[0,\tau_g]$. By using Corollary 3 and the periodicity of $g(t,x)$ we obtain that the stability of the periodic orbit is given by the sign of

$$\int_0^{\tau_g} \operatorname{div}\big(\big|g\big(t(s),\gamma\big(t(s)\big)\big)\big|X\big(t(s),\gamma\big(t(s)\big)\big)\big)\,ds.$$

Suppose that there are several periodic orbits totally contained in the region K. By using the above result all them have the same stability. Since the system has no critical points this is not possible and thus it can have at most one (hyperbolic) periodic orbit fully contained in K. $\square$

Now, we can state the result that gives information about the total number of periodic orbits of an Abel equation.

Theorem 5. *Consider the $\mathcal{C}^1$ Abel equation (8) on the strip $\mathcal{S}$. Suppose that there exist $w\in\mathbb{R}$ and a $\mathcal{C}^1$ function $f(t,x)$, 1-periodic in t and such that*

$$M_w(t,x) := \big\langle \nabla f(t,x), \big(1,h(t,x)\big)\big\rangle + w\, f(t,x)\operatorname{div}\big(1,h(t,x)\big)$$

does not change sign in the strip, vanishing only on a null measure Lebesgue set which is not invariant by the flow. Let K_1 denote the number of curves in $\{f=0\}$ joining $t=0$ and $t=1$, and let $K_2\leqslant K_1$ be the number of these curves which are invariant by the flow. Then the Abel equation has at most K_1+K_2+1 periodic orbits.

Moreover, all the orbits that are not contained in $\{f=0\}$ do not cut this set, are hyperbolic and their stability is given by the sign of $wf(t,x)M_w(t,x)$ on each of them.

Proof. Take the value w and the function f given in the statement of the theorem. Consider, for any $\mathcal{C}^1$ function g, the vector field associated to the Abel equation (7). Note that on $\{f=0\}$ the function $M_w(t,x)=\langle\nabla f,(1,h)\rangle$ does not change sign, which means that the flow of $(1,h(t,x))$ crosses the set $\{f=0\}$ only in one direction. Then, each periodic orbit either is contained in $\{f=0\}$ or does not intersect this set.

In order to bound the number of periodic orbits of the system which are not contained in $\{f=0\}$ consider in each connected component of $\mathcal{S}\setminus\{f=0\}$ the function $g(t,x)=|f(t,x)|^{1/w}$. If we compute now $\operatorname{div}(g(t,x)(1,h(t,x)))$ we get:

$$\operatorname{div}\big(g(t,x)\big(1,h(t,x)\big)\big) = \operatorname{sgn}(f)\frac{1}{w}|f|^{\frac{1}{w}-1}M_w(t,x).$$

In each connected component we can apply Corollary 4 and we get the upper bound stated above. □

Proposition 6. *Consider the differential equation*

$$\frac{dx}{dt} = f(t)P(x),$$

with f and P smooth functions. Assume that the equation $P(x) = 0$ has finitely many solutions, $x_1, x_2, \dots, x_n$. If $\int_0^1 f(t)\,dt = 0$, then all the solutions $x = x_i$ for $i = 1, \dots, n$, are centers; *otherwise, its only periodic orbits are $x = x_i$, for $i = 1, \dots, n$. Furthermore, in this later case, the simple zeros of $P(x)$ are hyperbolic periodic orbits of the differential equation.*

Proof. A solution $x = x(t)$ with initial condition $x(0) = \rho \neq x_i$, for $i = 1, \dots, n$ satisfies

$$\varphi\big(x(t)\big) := \int_{\rho}^{x(t)} \frac{du}{P(u)} = \int_0^t f(s)\,ds.$$

Since $\varphi'(x) = \frac{1}{P(x)}$, we have that $\varphi'(x) \neq 0$ if $\rho \neq x_i$ (because the solutions $x = x_i$ cannot be cut). This implies that $\varphi(x)$ is injective. Then, if $\int_0^1 f(t)\,dt = 0$, as $\varphi(\rho) = 0$ and $\varphi(x(1)) = 0$ we get $x(1) = \rho$ for any solution $x(t)$ with initial condition ρ, close enough to a periodic orbit $x = x_i$.

On the other hand, if $\int_0^1 f(t)\,dt \neq 0$, we get $\varphi(x(1)) - \varphi(x(0)) \neq 0$ and then, $x(1) \neq \rho$ if $\rho \neq x_i$ and the solution $x(t)$ is not periodic.

To prove the hyperbolicity of $x = x_i$ we have to compute the derivative of the Poincaré map Π defined by the flow between the two transversal sections $t = 0$ and $t = 1$ and see that it is not equal to 1. Following [9] we get that

$$\Pi'(x_i) = \exp\left(\int_0^1 \frac{\partial}{\partial x}\big(P(x)f(t)\big)\bigg|_{x=x_i} dt\right) = \exp\left(P'(x_i)\int_0^1 f(t)\,dt\right) \neq 1,$$

as we wanted to prove. □

We will also need some results referred to the case $C(t) \equiv 0$.

The next lemma is a straightforward consequence of the results of [1].

Lemma 7. *Consider the Abel equation* (1). *The solution $x = 0$ is a periodic orbit of multiplicity at least two, and the first necessary conditions to be a center are*

$$V_2 = \int_0^1 B(t)\,dt = 0, \qquad V_3 = \int_0^1 A(t)\,dt = 0 \quad \textit{and}$$

$$V_4 = \int_0^1 A(t)\left(\int_0^t B(s)\,ds\right) dt = 0.$$

Proposition 8. *Consider the Abel equation* (1). *If either* $A(t) \not\equiv 0$ *or* $B(t) \not\equiv 0$ *does not change sign in* $[0, 1]$ *then the equation has at most one non-zero periodic orbit and if it exists, it is hyperbolic.*

Proof. If either $A(t)$ or $B(t)$ does not change sign by using the results of [6] we have that Eq. (1) has at most three periodic orbits, taking into account their multiplicities. By Lemma 7 we know that the solution $x = 0$ is at least a double periodic orbit of the Abel equation. Hence we have proved that under our hypotheses (1) has at most one non-zero periodic orbit and that when it exists, it is hyperbolic. □

Remark 9. Note that if we want to study the number of periodic orbits of Eq. (1) when either $A(t)$ or $B(t)$ is a constant function the above two results solve completely the problem, giving the existence of at most one non-zero hyperbolic periodic orbit. Proposition 6 solves the case when the constant is zero, while Proposition 8 solves the case when it is not zero.

3. Lower bounds

In this section we construct Abel equations with two non-zero hyperbolic periodic orbits for the two families considered in this paper and we study when the solution $x = 0$ is a center.

3.1. Trigonometric case

Consider the Abel equation

$$\begin{aligned}\frac{dx}{dt} = A(t)x^3 + B(t)x^2 &= \big(a_0 + a_1\cos(2\pi t) + a_2\sin(2\pi t)\big)x^3 \\ &\quad + \big(b_0 + b_1\cos(2\pi t) + b_2\sin(2\pi t)\big)x^2.\end{aligned} \tag{9}$$

In this section we will see how to construct examples with two non-zero periodic orbits by using two different methods: bifurcating periodic orbits from $x = 0$ and studying perturbations of some centers inside the family.

3.1.1. Computation of the Lyapunov constants for $x = 0$

The first method we use to give a lower bound for the number of periodic orbits of Eq. (9) consists in computing how many periodic orbits can bifurcate from $x = 0$. For this purpose we compute the derivatives of the Poincaré map Π between $t = 0$ and $t = 1$, see [1] or [9]. For similarity with the planar case we will call these derivatives (modulus some non-zero multiplicative constants) the Lyapunov constants of $x = 0$. We have the following result.

Proposition 10. *Consider Eq.* (9). *The maximum number of non-zero periodic orbits that can bifurcate from* $x = 0$ *is two. Moreover, the solution* $x = 0$ *is a center if and only if* $b_0 = a_0 = a_2b_1 - a_1b_2 = 0$.

Proof. We compute the Lyapunov constants for $x = 0$ following Lemma 7. The first one is

$$V_2 = \int_0^1 B(t)\,dt = b_0.$$

If $V_2 \neq 0$, $x = 0$ is a semi-stable periodic orbit, i.e. if $b_0 > 0$ the orbits close to $x = 0$ with positive initial condition move away from it and the ones with negative initial condition approach it, while if $b_0 < 0$ the orbits with positive initial condition approach $x = 0$ and the ones with negative initial condition move away.

When $b_0 = 0$ we compute the next Lyapunov constant:

$$V_3 = \int_0^1 A(t)\,dt = a_0.$$

We get similar results as above but now considering a_0. If $a_0 = 0$, we compute another Lyapunov constant

$$V_4 = \int_0^1 A(t)\left(\int_0^t B(s)\,ds\right)dt = \frac{a_2 b_1 - a_1 b_2}{4\pi}.$$

Suppose that this quantity is positive, then if we choose $a_0 < 0$ and $b_0 > 0$ with $b_0 \ll |a_0| \ll a_2 b_1 - a_1 b_2$, by a degenerate Hopf bifurcation we can generate two periodic orbits from $x = 0$.

Let us prove now that if $V_2 = V_3 = V_4 = 0$, i.e. $a_0 = b_0 = a_2 b_1 - a_1 b_2 = 0$ then $x = 0$ is a center. If $b_1 = b_2 = 0$ we have $B(t) \equiv 0$ and, applying Proposition 6 we conclude that $x = 0$ is a center. Otherwise, suppose, for instance, that $b_2 \neq 0$. Then $a_1 = \frac{a_2 b_1}{b_2}$ and we have $A(t) = \frac{a_2}{b_2} B(t)$. Then, applying again Proposition 6 with $f(t) = B(t)$ and $P(x) = a_2 x^3 / b_2 + x^2$ we have again that $x = 0$ is a center, because $\int_0^1 B(t)\,dt = 0$. □

3.1.2. Perturbation of a center

The second method we use to produce examples with periodic orbits in Eq. (9) is the perturbation of a center inside this family.

Proposition 11. *Consider the equation*

$$\frac{dx}{dt} = 2\pi \tilde{b}_1 \cos(2\pi t) x^2 + \varepsilon\big[\big(\tilde{a}_0 + \tilde{a}_1 \cos(2\pi t) + \tilde{a}_2 \sin(2\pi t)\big)x^3 + \tilde{b}_0 x^2\big]$$

with $\tilde{b}_1 \neq 0$. Then, for ε small enough, at most two non-zero periodic orbits bifurcate from the continuous of periodic orbits existing when $\varepsilon = 0$. Furthermore, this upper bound can be reached and the periodic orbits obtained are hyperbolic.

Proof. The solutions of the above equation can be expanded in a small neighborhood of $\varepsilon = 0$ as

$$x_\varepsilon(t;\rho) = x_0(t;\rho) + \varepsilon S(t,\rho) + \varepsilon^2 R(t,\rho,\varepsilon),$$

where $S(0,\rho) = 0$ and $x_0(t;\rho)$ is the solution of the unperturbed equation given by

$$x_0(t;\rho) = \frac{\rho}{1 - \rho(\int_0^t B(s)\,ds)} = \frac{\rho}{1 - \rho \tilde{b}_1 \sin(2\pi t)}.$$

Set $W(\rho) := S(1,\rho)$. Following the ideas of [8] we know that the simple zeros of $W(\rho)$ will give rise to initial conditions of periodic orbits of the perturbed differential equation, which tend to these values when ε tends to zero. Doing some computations we get

$$\hat{W}(\rho) := \frac{W(\rho)}{\rho^2} = \int_0^1 \left(\tilde{b}_0 + \frac{\tilde{a}_0 + \tilde{a}_1\cos(2\pi t) + \tilde{a}_2\sin(2\pi t)}{1 - \tilde{b}_1\rho\sin(2\pi t)}\rho\right) dt,$$

which is well-defined in the region where the center exists, i.e. $|\tilde{b}_1\rho| < 1$. This fact induces to introduce the natural change of variable $\tilde{b}_1\rho = \sin(y)$, for $y \in (-\pi/2, \pi/2)$, for studying the non-zero zeros of $\hat{W}$. We obtain that

$$\begin{aligned}\hat{W}(\rho) &= \tilde{b}_0 + \rho\frac{\tilde{a}_0\sin(y) + \tilde{a}_2 - \tilde{a}_2\cos(y)}{\sin(y)\cos(y)}\\ &= \frac{\tilde{a}_0\sin(y) + (\tilde{b}_0\tilde{b}_1 - \tilde{a}_2)\cos(y) + \tilde{a}_2}{\tilde{b}_1\cos(y)}.\end{aligned}$$

Solving the equation $\hat{W}(\rho) = 0$ we get that it has in $(-\pi/2, \pi/2)$, at most two non-zero simple solutions, namely ρ_1 and ρ_2. These solutions will give rise, at most, to two non-zero periodic orbits of the perturbed differential equation, which can be easily seen that are hyperbolic because $\hat{W}'(\rho_i) \neq 0$, $i = 1, 2$. It is also clear that this upper bound can be reached. Consider for instance a system with $\tilde{a}_0 = 0$, $\tilde{b}_0\tilde{b}_1 - \tilde{a}_2 = 1$ and $\tilde{a}_2 = 3/4$. Thus the result follows. □

3.2. *Polynomial case*

In this subsection we consider the Abel equation (3):

$$\frac{dx}{dt} = A(t)x^3 + B(t)x^2 = \left(a_0 + a_1t^j + a_2t^k\right)x^3 + \left(b_0 + b_1t^j + b_2t^k\right)x^2.$$

As in the previous subsection we want to produce examples with as many non-zero periodic orbits as possible. As before we obtain examples with two non-zero periodic orbits. However here we only consider the method of Lyapunov constants, because the other method gives rise to tedious integrals and our impression is that again it does not produce more periodic orbits.

3.2.1. *Computation of the Lyapunov constants for $x = 0$*

We have the following result.

Proposition 12. *Consider Eq.* (3). *The maximum number of non-zero periodic orbits that can bifurcate from $x = 0$ is two. Moreover, the solution $x = 0$ is a center if and only if*

$$a_0 + \frac{a_1}{j+1} + \frac{a_2}{k+1} = b_0 + \frac{b_1}{j+1} + \frac{b_2}{k+1} = a_2b_1 - a_1b_2 = 0.$$

Proof. Following the same computations than in Proposition 10 we get that

$$V_2 = \int_0^1 B(t)\,dt = b_0 + \frac{b_1}{j+1} + \frac{b_2}{k+1}.$$

When $b_0 = -((k+1)b_1 + (j+1)b_2)/((j+1)(k+1))$,

$$V_3 = \int_0^1 A(t)\,dt = a_0 + \frac{a_1}{j+1} + \frac{a_2}{k+1}.$$

Finally when $V_2 = V_3 = 0$,

$$V_4 = \int_0^1 A(t)\left(\int_0^t B(s)\,ds\right)dt = \frac{jk(k-j)(a_2b_1 - a_1b_2)}{2(1+j)(2+j)(1+k)(2+k)(2+j+k)}.$$

If $V_4 > 0$ we choose $a_0 = -\frac{(k+1)a_1+(j+1)a_2}{(j+1)(k+1)} - \mu$ and $b_0 = -\frac{(k+1)b_1+(j+1)b_2}{(j+1)(k+1)} + \lambda$ with $\lambda, \mu > 0$, in such a way that $\lambda \ll \mu \ll a_2b_1 - a_1b_2$. Then, by a degenerate Hopf bifurcation we can generate two non-zero periodic orbits from $x = 0$. Finally when $V_2 = V_3 = V_4 = 0$, the fact that the origin is a center follows by Proposition 6. □

4. Proofs of the main results

Proof of Theorem A. In our proof we do not care about the case $ab = 0$ because it follows easier than when $ab \neq 0$. Furthermore, when $ab = 0$ the results also follows from [6], see Proposition 8.

We start by proving that (1) has at most three non-zero periodic orbits.

We apply Theorem 5 with $f(x) = x^2(bx - a)$ and $w = -1$. The function $M_{-1}(t, x)$ is:

$$M_{-1}(t, x) = x^4\big(aA(t) + bB(t)\big),$$

and it never vanishes identically. Thus by applying Theorem 5 we know that two types of periodic orbits can exist: the ones contained in the set $\{f = 0\}$, i.e. $x = a/b$ and $x = 0$, and the ones that are contained in $\mathcal{S} \setminus \{f = 0\}$. Since $aA(t) + bB(t) \not\equiv 0$ then $x = a/b$ is not a periodic orbit.

Thus the maximum number of non-zero periodic orbits is three, two living in the same semi-strip where the curve $x = a/b$ is, one bigger than this curve and the other one smaller, and a third one in the other half-strip. The hyperbolicity of these periodic orbits is also given by Theorem 5.

To prove that in fact there is at most one non-zero periodic orbit we will compare our differential equation with two simpler ones for which we know their phase portraits. More precisely we write the Abel equation (1) in the following two ways

$$\frac{dx}{dt} = A(t)x^2\left(x - \frac{a}{b}\right) + \frac{aA(t) + bB(t)}{b}x^2,$$
$$\frac{dx}{dt} = -\frac{b}{a}B(t)x^2\left(x - \frac{a}{b}\right) + \frac{aA(t) + bB(t)}{a}x^3.$$

By changing the sign of t if necessary it is not restrictive to assume that $h(t) := (aA(t) + bB(t))/b \geqslant 0$. The above equations write as

$$\frac{dx}{dt} = A(t)x^2\left(x - \frac{a}{b}\right) + h(t)x^2, \tag{10}$$

$$\frac{dx}{dt} = -\frac{b}{a}B(t)x^2\left(x - \frac{a}{b}\right) + \frac{b}{a}h(t)x^3. \tag{11}$$

We start comparing (10) with

$$\frac{dx}{dt} = A(t)\,x^2\left(x - \frac{a}{b}\right). \tag{12}$$

The global phase portrait of Eq. (12) is given by Proposition 6. When $\int_0^1 A(t)\,dt = 0$ the equation has a center at $x = 0$ and $x = a/b$ and, since $h(t)x^2 \geqslant 0$, all the solutions of (12), except $x = 0$, are curves without contact for the flow of (10). Thus, $x = 0$ is its only periodic orbit.

If $\int_0^1 A(t)\,dt \neq 0$ then $x = 0$ is a double periodic orbit of (12) and $x = a/b$ is a hyperbolic one and the behavior of the other solutions is determined by the sign of $-a/b\int_0^1 A(t)\,dt$. Assume firstly that $-a/b\int_0^1 A(t)\,dt > 0$ and that $a/b > 0$ (the case $a/b < 0$ follows similarly). Under these inequalities, the solutions of (10) starting below $x = 0$ approach this periodic orbit and the solutions starting above $x = 0$ approach $x = a/b$. In other words all solutions of (12), $x(t; x_0)$ with $x(0; x_0) = x_0$ where $x_0 < a/b$ and $x_0 \neq 0$ satisfy that $x(1; x_0) > x_0$. Let $\bar{x}(t; x_0)$ be the solution of (10) starting also at x_0. Note that it satisfies the differential inequality

$$\dot{x} = A(t)x^2\left(x - \frac{a}{b}\right) + h(t)x^2 \geqslant A(t)x^2\left(x - \frac{a}{b}\right),$$

and thus $\bar{x}(1; x_0) > x(1; x_0) > x_0$ for all $x_0 \neq 0$ bellow $x = a/b$. Hence, there is no non-zero periodic orbit below $x = a/b$ and we are done, because we already know that in the region $x > a/b$ there is at most one periodic orbit.

If $-a/b\int_0^1 A(t)\,dt < 0$ and $a/b > 0$ the same reasoning can be applied to the region $x > a/b$ and there is no periodic orbit there. To study the region $x < a/b$ we distinguish three cases according to the stability of $x = 0$ for (1), which is given by the sign of $\int_0^1 B(t)\,dt$.

If $\int_0^1 B(t)\,dt = 0$, arguing as in the case $\int_0^1 A(t)\,dt = 0$ we can prove that $x = 0$ is the only periodic orbit. If $\int_0^1 B(t) > 0$ there is no periodic orbit in $0 < x < a/b$. The reason is that in this strip there is at most a periodic orbit which, if exists, is hyperbolic, and the flow near the boundaries of this region implies that if a periodic orbit would exist it should have even multiplicity. Thus, the periodic orbit, if exists, it is unique and located in the region $x < 0$. Finally, if $\int_0^1 B(t)\,dt < 0$ the comparison of the flow of (10) and (12) and the previous results prove the existence of exactly one periodic orbit with initial condition between 0 and a/b. The reason is again the sense of the flow in the boundaries of this region. In order to prove that there is no periodic orbit in $x < 0$, we compare (10) with

$$\frac{dx}{dt} = -\frac{b}{a}B(t)x^2\left(x - \frac{a}{b}\right). \tag{13}$$

Since precisely in the region $x < 0$ we have $h(t)x^3 \leqslant 0$, by using similar arguments than before we prove that in this region there is no periodic orbit. Thus the theorem follows. □

Remark 13. The proof of Theorem A also helps to locate the non-zero periodic orbit and in some cases to prove its existence. Collecting all the above results when $(aA(t) + bB(t))/b \geqslant 0$ and $a/b > 0$ we obtain that, if the non-zero periodic orbit exists, it is located in:

(i) the region $x > a/b$, when $\int_0^1 A(t)\,dt < 0$,
(ii) the region $x < 0$, when $\int_0^1 A(t)\,dt > 0$ and $\int_0^1 B(t)\,dt > 0$,
(iii) the region $0 < x < a/b$ when $\int_0^1 A(t)\,dt > 0$ and $\int_0^1 B(t)\,dt < 0$. Furthermore in this case it always exists.

Other signs of $(aA(t) + bB(t))/b$ and a/b can be studied similarly.

Proof of Theorem B. (a) Follows from Proposition 10.

(b) By applying Theorem A with $a = -1$ and $b = m$ we get that

$$\begin{aligned} M_{-1}(t,x) &= x^4\bigl(mB(t) - A(t)\bigr) \\ &= x^4\bigl(b_0m - a_0 + (b_1m - a_1)\cos(2\pi t) + (b_2m - a_2)\sin(2\pi t)\bigr), \end{aligned}$$

and never vanishes identically. Note that if we can find an m such that $(b_0m - a_0)^2 \geqslant (b_1m - a_1)^2 + (b_2m - a_2)^2$ then M_{-1} will not change sign. This m always exists if one of the three conditions of the theorem is satisfied. Thus by applying again Theorem A we know that there is at most one non-zero periodic orbit. The hyperbolicity of this periodic orbit is also given by the same theorem.

(c) The result follows by using either Propositions 10 or 11. □

Remark 14. Notice that following the proof of Theorem B we obtain that for each m for which $(b_0m - a_0)^2 \geqslant (b_1m - a_1)^2 + (b_2m - a_2)^2$, the line $x = -1/m$ as a curve without contact for the flow of the differential equation. Taking all the curves together we obtain a kind of Lyapunov function that helps to locate the regions where periodic orbits can live. For instance if $a_0^2 > a_1^2 + a_2^2$ or $b_0^2 > b_1^2 + b_2^2$, and $(a_2b_0 - a_0b_2)^2 + (a_0b_1 - a_1b_0)^2 < (a_2b_1 - a_1b_2)^2$ then all the non-zero values of m are allowed in the proof of the above theorem and we can show that there is no non-zero periodic orbit.

Proof of Theorem C. (a) Follows from Proposition 12.

(b) The proof follows in a very similar way than the proof of (b) of Theorem B. In order to apply Theorem A we set $a = -b_2$ and $b = a_2$. Thus,

$$M_{-1}(t,x) = x^4\bigl(a_2B(t) - b_2A(t)\bigr) = x^4\bigl(a_2b_0 - a_0b_2 + (a_2b_1 - a_1b_2)t^j\bigr),$$

and, if condition (1) is satisfied, $M_{-1}(t,x)$ has a definite sign in all the strip.

To study the differential equation under condition (2) we can consider $a = -b_1$ and $b = a_1$. Then the function $M_{-1}(t,x)$ is given by

$$M_{-1}(t,x) = x^4\bigl(a_1b_0 - a_0b_1 + (a_1b_2 - a_2b_1)t^k\bigr).$$

Finally, when condition (3) is considered, by using $a = -b_0$ and $b = a_0$, we get that

$$M_{-1}(t,x) = x^4 t^j \big(a_0 b_1 - a_1 b_0 + (a_0 b_2 - a_2 b_0) t^{k-j}\big).$$

(c) The result follows by using Proposition 12. □

5. The general Abel equation

Regarding the general Abel equation (2):

$$\frac{dx}{dt} = A(t)x^3 + B(t)x^2 + C(t)x,$$

we have proved the following two extensions of Theorem A:

Theorem 15. *Consider the general Abel equation* (2). *Assume that* $\int_0^1 C(t)\,dt = 0$ *and that there exist two real numbers* a *and* b *such that*

$$aA(t)\exp\left(\int_0^t C(s)\,ds\right) + bB(t)$$

does not vanish identically and does not change sign for all $t \in [0,1]$. *Then it has at most one non-zero periodic orbit. Furthermore, when this periodic orbit exists, it is hyperbolic.*

Proof. If $\int_0^1 C(t)\,dt = 0$, the well-known change of variables

$$y = x\exp\left(-\int_0^t C(s)\,ds\right)$$

transforms Eq. (2) into

$$\frac{dy}{dt} = A(t)\exp\left(2\int_0^t C(s)\,ds\right)y^3 + B(t)\exp\left(\int_0^t C(s)\,ds\right)y^2.$$

Furthermore this change sends periodic orbits of (2) into periodic orbits of the above equation. By applying Theorem A to this equation the result follows. □

Theorem 16. *Consider Eq.* (2). *Assume that there exist three real numbers* a, b *and* c *such that* $aA(t) + bB(t)$ *is not identically zero and does not change sign for all* $t \in [0,1]$, *and that* $(bC(t) - cA(t))^2 + (aA(t) + bB(t))(cB(t) + aC(t)) < 0$ *for all* $t \in [0,1]$. *Then it has at most four non-zero periodic orbits.*

Proof. As in the proof of Theorem A we can only consider the case $ab \neq 0$ and we apply Theorem 5 with $f(x) = bx^3 - ax^2 + cx$ and $w = -1$. The function $M_{-1}(t,x)$ is:

$$M_{-1}(t,x) = x^2\big(\big(aA(t) + bB(t)\big)x^2 + 2\big(bC(t) - cA(t)\big)x - \big(cB(t) + aC(t)\big)\big).$$

If the two conditions of the statement of the theorem are satisfied then this function does not change sign, and by using Theorem 5, we know that only two types of periodic orbits can exist: the ones contained in the set $\{f = 0\}$, i.e. $x^{\pm} := (a \pm \sqrt{a^2 - 4bc})/(2b)$ and $x = 0$, and the ones that are contained in $\mathcal{S} \setminus \{f = 0\}$. Furthermore, there is at most one periodic orbit of this type in each connected component of $\mathcal{S} \setminus \{f = 0\}$.

Thus, when $\Delta := a^2 - 4bc < 0$ ($\Delta = 0$, $\Delta > 0$, respectively) Eq. (2) has at most 2 (4, 6, respectively) non-zero periodic orbits. Hence, when $a^2 - 4bc \leqslant 0$ we are done. When $a^2 - 4bc > 0$ and none of the lines $x = x^{\pm}$ is a periodic orbit the result also follows. Finally assume that one of these lines is a periodic orbit. In this situation it is easy to see that the other one cannot be a periodic orbit of the equation. Then we have proved that the maximum number of non-zero periodic orbits is five: the one given by the invariant line, say $x = x^+ \neq 0$, and the other four contained in $\mathcal{S} \setminus \{f = 0\}$. To end the proof we will see that in this case there are at most three periodic orbits in $\mathcal{S} \setminus \{f = 0\}$. If $x = x^+$ is a periodic orbit of (2) then

$$0 \equiv \frac{dx}{dt}\bigg|_{x=x^+} = x^+\left(A(t)x^+\left(x^+ - \frac{a}{b}\right) + C(t) + x^+h(t)\right),$$

where $h(t) = (aA(t) + bB(t))/b$. Then $C(t) = A(t)x^+(\frac{a}{b} - x^+) - x^+h(t)$ and Eq. (2) writes as

$$\frac{dx}{dt} = x\big(x - x^+\big)\big(A(t)\big(x - x^-\big) + h(t)\big).$$

We will compare the solutions of the previous equation with the solutions of

$$\frac{dx}{dt} = A(t)x\big(x - x^+\big)\big(x - x^-\big).$$

By using Proposition 6 we know that when $\int_0^1 A(t)\,dt = 0$ this equation has centers at $x = 0$, $x = x^+$ and $x = x^-$. When $\int_0^1 A(t)\,dt \neq 0$, $x = 0$, $x = x^+$ and $x = x^-$ are the only periodic orbits. Following the same reasoning than in the proof of Theorem A, we can distinguish the cases according to the sign of several involved functions, obtaining in all cases that there is no periodic orbit in one of the four connected components of the set $\mathcal{S} \setminus \{f = 0\}$. Thus the result follows. $\square$

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